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Loop Groups, Anyons and the Calogero–Sutherland Model

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Abstract: The positive energy representations of the loop group of U(1) are used to construct a boson-anyon correspondence. We compute all the correlation functions of our anyon fields and study an anyonic W-algebra of unbounded operators with a common dense domain. This algebra contains an operator with peculiar exchange relations with the anyon fields. This operator can be interpreted as a second quantized Calogero–Sutherland (CS) Hamiltonian and may be used to solve the CS model. In particular, we inductively construct all eigenfunctions of the CS model from anyon correlation functions, for all particle numbers and positive couplings.

1. Introduction

The viewpoint of Graeme Segal [PS, SeW] on integrable systems links the infinite dimensional Grassmanian approach of Sato [S] with the representation theory of loop groups. These two points of view overlap in the study of two dimensional quantum field theories. In the Sato approach, as in much of the physics literature, quantum field theory is regarded as an algebraic theory in which the usual Hilbert space formalism is absent. The Segal approach on the other hand deals with positive energy representations of loop groups in Hilbert spaces. Reconciling these points of view can be quite difficult although this has been done for many cases (see for example [CR, CHMS, BMT]). One way of thinking about the Segal approach is that it revolves around a Hilbert space definition of vertex operators. The algebraic approach to vertex operators is much studied in connection with Kac-Moody algebras [K, F] and may be regarded as the Lie algebraic version of the loop group projective representation theory. These Segal vertex operators arise from a boson field theory and were previously studied in a formal way in [Sk, C, M] and made more precise in [StW, DFZ]). In this approach one regularizes the vertex operators so that they are proportional to operators representing loop group elements and then, after taking an appropriate limit, one finds that they generate fermions in some cases (the boson-fermion correspondence) and operators forming a Kac-Moody algebra in others [PS, Se, CR, CHu], depending on the precise form of the cocycle in the loop group projective representation.

We may summarize the present paper as enlarging the loop group representation theory to encompass a boson-anyon correspondence. Our results extend those of the previous paragraph in that we construct, from a certain positive energy loop group representation, Segal-type vertex operators on a Hilbert space which have, as their limits, anyon field operators. These anyon field operators applied to the vacuum, or cyclic vector, give new vectors in the Hilbert space which can be interpreted as anyon states. Each N-particle anyon sector carries a representation of the braid group. The construction builds in fractional statistics from the outset, the precise statistics depending on the choice of anyon vertex operator.

The idea of using a vertex operator construction to obtain particles with anyon type statistics is not new, see for example [Kl] and more recently [AMOS1, AMOS2, I, H, MS] and references therein. However the vertex operators described in these more recent references are not defined on the Fermion Fock space as limits of implementors of fermion gauge transformations. In other words they do not come from loop group elements. Indeed it is difficult to give a precise meaning to them at all and we do not attempt to do so here. Our vertex operators can be seen to have similar formal properties to those appearing in the papers mentioned, but are well defined in terms of positive energy representations of loop groups in the sense of [PS].

The benefits of our approach are the following. First there is a quantum Hamiltonian acting on the anyon states. This we believe resolves a long standing difficulty in the study of anyons in that it provides a basis for models incorporating interactions. Second we obtain a unifying view of a number of interesting ideas that have emerged in recent times in the physics literature. The most important of these is the connection with the Calogero–Sutherland (CS) model [AMOS1, AMOS2, I, MS] (see also [H, HLV, BHKV, P]). Specifically we find that n-point anyon correlation functions provide useful building blocks for solutions to the CS system. Comparing with the known solutions of the CS system [Fo2] we find that Jack polynomials [St] may be expressed in terms of anyon correlation functions. (Similar relations were previously obtained by different methods in [Fo1].)

From this point of view the anyon Hamiltonian is a second quantized CS Hamiltonian. The final connection we make is with W-algebras, again a connection which has been known from other approaches for some time [AMOS1, AMOS2, I, MS]. In this paper we do not recover the full import of the W-algebra connection in the anyon case. This is a matter we intend to develop more fully elsewhere. However we do construct that part of the W-algebra that we need as an algebra of unbounded operators with a common dense domain. This suffices for our purposes, namely the construction of an anyon Hamiltonian, constructing the CS model solutions as anyon correlation functions, obtaining the link with Jack polynomials, and finding the algebraic relations of the Hamiltonian with the anyon fields.

2. Summary

This paper contains a number of technical sections. In order to make the results accessible we present a summary here. At the same time we take the opportunity to introduce some of our notation. However, the reader will need to take some notation on trust and refer to later sections for the details.

We work on an interval $S_L = [-L/2, L/2]$ which we will think of as a circle of circumference L. We let $P_\pm$ be the spectral projections of $-i\frac{\partial}{\partial x}$ regarded as a self adjoint operator on a dense domain in $L^2(S_L)$. We let $\mathcal{F}$ denote the free fermion Fock space over $L^2(S_L)$. We choose the usual positive energy condition that the fermion fields are in a Fock representation of the algebra of the canonical anticommutation relations defined by P_-. This means the fermion fields $\{\psi(f), \psi(g)^* \mid f, g \in L^2(S_L)\}$ satisfy

$$\langle \Omega, \psi(f)^* \psi(g)\Omega \rangle_{\mathcal{F}} = \langle g, P_- f \rangle_{L^2(S_L)}, \tag{1}$$

where Ω is the vacuum or cyclic vector in $\mathcal{F}$. We let Q denote the Fermion charge operator on $\mathcal{F}$, and R a unitary charge shift operator on $\mathcal{F}$ satisfying $R^{-1}QR = Q + I$ (the precise choice for R will be explained later).

We will construct regularized anyon field operators $\phi^\nu_\varepsilon(x)$, where $\nu \in \mathbb{R}$ is a parameter determining the statistics, $x \in S_L$, and $\varepsilon > 0$ is a regularization parameter. For positive ε the operator $\phi^\nu_\varepsilon(x)$ is proportional to a unitary operator on $\mathcal{F}$ which represents a certain U(1) valued loop on S_L. These operators are not periodic but obey (the parameter ν_0 will be explained below),

$$\phi^\nu_\varepsilon(x + L) = e^{-i\pi\nu\nu_0 Q} \phi^\nu_\varepsilon(x) e^{-i\pi\nu\nu_0 Q},$$

and in the limit as $\varepsilon \downarrow 0$ they converge to operator valued distributions $\phi^\nu(x)$ satisfying

$$\phi^\nu(x)\phi^{\nu'}(y) = e^{-i\pi\nu\nu' \mathrm{sgn}(x-y)} \phi^{\nu'}(y)\phi^\nu(x) \tag{2}$$

for $x \neq y$. In particular for $p \in \Lambda^* = \{\frac{2\pi}{L} n | n \in \mathbb{Z}\}$, the formula

$$\hat{\phi}^\nu(p) = \lim_{\varepsilon\downarrow 0} \int_{-L/2}^{L/2} dx\, e^{ipx} e^{i\pi\nu\nu_0 Qx/L} \phi^\nu_\varepsilon(x) e^{i\pi\nu\nu_0 Qx/L} \tag{3}$$

is a well-defined operator on $\mathcal{F}$ (Proposition 1). Note that we have to insert factors to compensate for the non-periodicity of $\phi^\nu_\varepsilon(x)$ before Fourier transformation. We also find that the statistics parameters ν, ν' for which Eq. (2) holds cannot be arbitrary but have to be integer multiples of some fixed (arbitrary) number $\nu_0 > 0$. (If one is only interested in a single species of anyons one can choose $\nu_0 = |\nu|$.)

A main focus is on the correlation functions of the anyon fields. These are distributions defined by taking the limit as $\varepsilon_j \downarrow 0$ of

$$C^{\nu_1,\dots,\nu_N}_{\varepsilon_1,\dots,\varepsilon_N}(\nu_0, w_1, w_2|y_1, \dots, y_N) := \left\langle \Omega, R^{w_1} \phi^{\nu_1}_{\varepsilon_1}(x_1) \cdots \phi^{\nu_N}_{\varepsilon_N}(x_N) R^{w_2} \Omega \right\rangle \tag{4}$$

for $x_j \in S_L$, $\nu_j/\nu_0, w_j \in \mathbb{Z}$ (for fixed ν_0). Using general results for implementors of U(1) loops we obtain

$$C^{\nu_1,\dots,\nu_N}_{\varepsilon_1,\dots,\varepsilon_N}(\nu_0, w_1, w_2|y_1, \dots, y_N) = \delta_{w_1+w_2+(\nu_1+\ldots+\nu_N)/\nu_0, 0}$$
$$\times e^{i\pi(w_1-w_2)\nu_0(\nu_1 x_1 + \cdots \nu_N x_N)/L} \prod_{j=1}^N \prod_{k=j+1}^N b(x_j - x_k; \varepsilon_j + \varepsilon_k)^{\nu_j\nu_k} \tag{5}$$

with

$$b(x, \varepsilon) := \left(e^{-i\frac{\pi}{L}x} - e^{-\frac{2\pi}{L}\varepsilon} e^{i\frac{\pi}{L}x} \right) = -2ie^{-\pi\varepsilon/L} \sin\frac{\pi}{L}(x + i\varepsilon). \tag{6}$$

The reason for studying these correlation functions is the connection with the Calogero–Sutherland (CS) Hamiltonian [Su]. This is defined on the set of functions $f \in C^2(S_L^N;\mathbb{C})$ which are zero on $\{(x_1,\dots,x_N)\in S_L^N | x_j = x_k \text{ for some } k\neq j \text{ and/or } x_j = \pm L/2\}$,

$$H_{N,\beta} = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{(\frac{\pi}{L})^2\beta(\beta-1)}{\sin^2\frac{\pi}{L}(x_j-x_k)}, \tag{7}$$

and which extends to a self-adjoint operator on $L^2(S_L^N)$.[1]

We will prove that the eigenfunctions and spectrum of this Hamiltonian can be obtained from anyon correlation functions, namely as finite linear combinations of functions

$$f_{\nu,N}(\mathbf{n}|\mathbf{x}) := \lim_{\varepsilon\downarrow 0}\left\langle \Omega, \hat\phi^\nu(\frac{2\pi}{L}n_N)^*\cdots\hat\phi^\nu(\frac{2\pi}{L}n_1)^*\phi^\nu_\varepsilon(x_1)\cdots\phi^\nu_\varepsilon(x_N)\Omega\right\rangle, \tag{8}$$

where $n_j\in\mathbb{N}_0$ (Theorem 3). We will obtain these results by constructing a self-adjoint operator $\mathcal{H}^{\nu,3}$ which can be regarded as a "second quantization" of the CS Hamiltonian Eq. (7) for $\beta=\nu^2$: it obeys the relations

$$\mathcal{H}^{\nu,3}\phi^\nu_\varepsilon(x_1)\cdots\phi^\nu_\varepsilon(x_N)\Omega \simeq H_{N,\nu^2}\phi^\nu_\varepsilon(x_1)\cdots\phi^\nu_\varepsilon(x_N)\Omega,$$

where "$\simeq$" mean "equal in the limit $\varepsilon\downarrow 0$" (see Theorem 2 for details). We obtain $\mathcal{H}^{\nu,3}$ by arguing by analogy with the well known W-algebra associated with fermions. Using analogous formulae we construct the first few generators $\mathcal{H}^{\nu,s}$, $s=1,2,3$, of an anyon W-algebra. Understanding the complete anyon W-algebra is a problem we leave for a further investigation.

This main result implies explicit formulas for the eigenvalues and a simple algorithm to construct eigenvectors $\Psi_{\nu,N}(\mathbf{n})$ of $\mathcal{H}^{\nu,2}$ as finite linear combinations of vectors

$$\eta_{\nu,N}(\mathbf{n}) = \hat\phi^\nu(\frac{2\pi}{L}n_1)\cdots\hat\phi^\nu(\frac{2\pi}{L}n_N)\Omega, \quad \mathbf{n}=(n_1,\dots,n_N)\in\mathbb{N}_0^N. \tag{9}$$

These vectors Eq. (9) can be naturally interpreted as N-anyon states with anyon momenta $p_j=\frac{2\pi}{L}n_j$. Using these relations we can compute

$$\left\langle \Psi_{\nu,N}(\mathbf{n}), \mathcal{H}^{\nu,3}\phi^\nu_\varepsilon(x_1)\cdots\phi^\nu_\varepsilon(x_N)\Omega\right\rangle$$

in two different ways, and in the limit $\varepsilon\downarrow 0$ we obtain functions (of the variables $(x_1,\dots,x_N)$) in $L^2(S_L^N)$ which are the promised eigenfunctions of H_{N,ν^2} (Theorem 3).

In the last subsection we observe that we recover the known spectrum of the CS Hamiltonian. Comparing with the known solutions of the CS model [Fo2], we can establish the relationship between the eigenfunctions of Theorem 3 and the Jack polynomials.

[1] Since Eq. (7) obviously is a positive symmetric operator, this follows e.g. from Theorem X.23 in Ref. [RS2] (the Friedrich's extension). Our approach will lead to a particular self-adjoint extension which is related to the standard one [Su] in a simple manner.

3. Preliminaries

The subsequent discussion relies on some standard material which is summarized in this section. We will follow essentially the treatment in [CHu, PS, CR].

3.1. Notation. We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the positive and non-negative integers, respectively. Let

$$\Lambda^* = \{p = \frac{2\pi}{L}n \quad | \quad n \in \mathbb{Z}\} \tag{10}$$

and

$$\Lambda_0^* = \{k = \frac{2\pi}{L}(n + \frac{1}{2}) \quad | \quad n \in \mathbb{Z}\}. \tag{11}$$

Our underlying Hilbert space for the fermions we take to be $L^2(S_L) \cong \ell^2(\Lambda_0^*)$. These are identified via the Fourier transform defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{L/2} dx f(x) e^{-ikx} \tag{12}$$

for $k \in \Lambda_0^*$. An orthogonal basis of $L^2(S_L)$ is provided by the functions

$$e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k \in \Lambda_0^*, \tag{13}$$

and then we have

$$f = \frac{2\pi}{L} \sum_k \hat{f}(k) e_k.$$

The spectral projection P_- corresponding to the negative eigenvalues of $\frac{1}{i}\frac{\partial}{\partial x}$ is defined as $\widehat{(P_- f)}(k) = \hat{f}(k)$ for $k < 0$ and = 0 otherwise. We also use $P_+ = I - P_-$.

3.2. Quasi-free representations of the CAR algebra. Let $\{a(f), a(g)^* \mid f, g \in L^2(S_L)\}$ be the usual generators of the fermion field algebra over $L^2(S_L)$, satisfying the canonical anticommutation relations (CAR)

$$a(f)a(g) + a(g)a(f) = 0, \quad a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle_{L^2(S_L)} I. \tag{14}$$

In the representation π_{P_-} of this algebra determined by the projection P_- we write $\psi(f) = \pi_{P_-}(a(f))$. If Ω denotes the cyclic (or vacuum) vector in the Fock space $\mathcal{F}$ on which π_{P_-} acts then this representation is specified by the following conditions,

$$\psi(P_+ f)\Omega = 0 = \psi^*(P_- f)\Omega. \tag{15}$$

We also use the notation

$$\hat{\psi}^{(*)}(k) = \psi^{(*)}(e_k), \quad k \in \Lambda_0^*. \tag{16}$$

3.3. Wedge representation of the loop group. Each unitary operator U on $L^2(S_L)$, with $P_\pm U P_\mp$ Hilbert–Schmidt, defines an "implementer" $\Gamma(U)$, on the Fock space $\mathcal{F}$ satisfying

$$\Gamma(U)\psi(f)\Gamma(U)^{-1} = \psi(Uf). \tag{17}$$

Of particular interest is the representation of the smooth loop group $\mathcal{G} = C^\infty(S_L;U(1))$ of $U(1)$ by implementors of the unitaries $U(\varphi)$ acting on $L^2(S_L)$. These are defined for $\varphi \in \mathcal{G}$ by

$$U(\varphi)f = \varphi f, \quad f \in L^2(S_L). \tag{18}$$

Then Γ gives a projective representation of $\mathcal{G}$ on $\mathcal{F}$. Writing $\Gamma(\varphi)$ for $\Gamma(U(\varphi))$ we may choose

$$\Gamma(\varphi)^* = \Gamma(\varphi^*) \tag{19}$$

and we have

$$\Gamma(\varphi)\Gamma(\varphi') = \sigma(\varphi,\varphi')\Gamma(\varphi\varphi'), \tag{20}$$

where $\sigma(\varphi,\varphi')$ is some $U(1)$ valued group two-cocycle on $\mathcal{G}$. We will determine this cocycle next.

The choice of phase of $\Gamma(\varphi)$ is important for giving an exact formula for σ. For those $\varphi = e^{i\alpha}$, with $\alpha \in Lie\mathcal{G} := C^\infty(S_L;\mathbb{R})$, the map $r \to \Gamma(e^{ir\alpha})$ is required to be a one parameter group such that the generator $d\Gamma(\alpha)$ of this group satisfies $\langle \Omega, d\Gamma(\alpha)\Omega\rangle = 0$. Then we have

$$[d\Gamma(\alpha), \psi(g)^*] = \psi(\alpha g)^* \tag{21}$$

and a standard calculation [CHu, PS, CR] gives

$$[d\Gamma(\alpha_1), d\Gamma(\alpha_2)] = is(\alpha_1,\alpha_2)I \tag{22}$$

with the Lie algebra two-cocycle

$$s(\alpha_1,\alpha_2) = \frac{1}{4\pi}\int_{-L/2}^{L/2} dx \Big(\frac{d\alpha_1(x)}{dx}\alpha_2(x) - \alpha_1(x)\frac{d\alpha_2(x)}{dx}\Big). \tag{23}$$

Hence the

$$\Gamma(e^{i\alpha}) = e^{id\Gamma(\alpha)} \tag{24}$$

are Weyl operators satisfying Eq. (20) with $\sigma(e^{i\alpha_1}, e^{i\alpha_2}) = e^{-is(\alpha_1,\alpha_2)/2}$.

We will also use $d\Gamma(\alpha)$ for complex valued α. These are naturally defined by linearity,

$$d\Gamma(\alpha_1 + i\alpha_2) = d\Gamma(\alpha_1) + id\Gamma(\alpha_2) \quad \alpha_{1,2} \in C^\infty(S_L;\mathbb{R}). \tag{25}$$

Then

$$d\Gamma(\alpha)^* = d\Gamma(\alpha^*) \tag{26}$$

(we use the same symbol $*$ for Hilbert space adjoints and complex conjugation), and Eqs. (22) and (23) extend to $C^\infty(S_L;\mathbb{C})$ so that s defines a complex bilinear form in an obvious way.

Here a technical remark is in order. The operators $d\Gamma(\alpha)$, $\alpha \in C^\infty(S_L;\mathbb{C})$, are all unbounded. However, there is a common, dense, domain $\mathcal{D}$ which is left invariant by all operators $\Gamma(\varphi)$, $\varphi \in \mathcal{G}$ (this is discussed in more detail in Appendix B). Thus Eqs. (21), (22), (25) and similar equations below are all well-defined on $\mathcal{D}$. We also note that all vectors in $\mathcal{D}$ are analytic for all the operators $d\Gamma(\alpha)$, $\alpha \in C^\infty(S_L;\mathbb{C})$ (see e.g. [CR]).

It is convenient to decompose loops into their positive, negative and zero Fourier components,

$$\alpha(x) = \alpha^+(x) + \alpha^-(x) + \bar{\alpha};\ \alpha^{\pm}(x) = \frac{1}{L}\sum_{\pm p>0}\hat{\alpha}(p)e^{ipx}, \quad \bar{\alpha} = \frac{1}{L}\hat{\alpha}(0) \tag{27}$$

where

$$\hat{\alpha}(p) = \int_{-L/2}^{L/2} dx\, \alpha(x)e^{-ipx} \quad p \in \Lambda^*. \tag{28}$$

Then

$$d\Gamma(\alpha) = d\Gamma(\alpha^+) + d\Gamma(\alpha^-) + \bar{\alpha}Q \tag{29}$$

with $Q = d\Gamma(I)$. Note that

$$d\Gamma(\alpha^-)\Omega = d\Gamma(\alpha^+)^*\Omega = Q\Omega = 0 \tag{30}$$

(highest weight condition) implying

$$\left\langle \Omega, d\Gamma(\alpha_1)d\Gamma(\alpha_2)\Omega\right\rangle = \left\langle \Omega, d\Gamma(\alpha_1^-)d\Gamma(\alpha_2^+)\Omega\right\rangle = is(\alpha_1^-, \alpha_2^+). \tag{31}$$

We also have $is(\alpha^-, \alpha^+) \geq 0$ which can be also easily be seen from the explicit formula for s,

$$is(\alpha_1, \alpha_2) = \sum_{p\in\Lambda^*} \frac{p}{2\pi L}\hat{\alpha}_1(-p)\hat{\alpha}_2(p). \tag{32}$$

Standard arguments now give us (for α real valued)

$$\left\langle \Omega, \Gamma(e^{i\alpha})\Omega\right\rangle = e^{-is(\alpha^-,\alpha^+)}. \tag{33}$$

We also need $R = \Gamma(\phi_1)$ which implements the operator $U(\phi_1)$, where $\phi_1(x) = e^{2\pi ix/L}$ (for an explicit construction of $\Gamma(\phi_1)$ see e.g. [R]). The phase of this unitary operator will be fixed latter. Notice that

$$R^{-1}d\Gamma(\alpha)R = d\Gamma(\alpha) + \bar{\alpha}I. \tag{34}$$

(This will be explained in more detail in Appendix A.)

General loops in $\mathcal{G}$ are of the form $\varphi = e^{if}$ with

$$f(x) = w\frac{2\pi}{L}x + \alpha(x) \tag{35}$$

with periodic α and integer $w = [f(L/2) - f(-L/2)]/2\pi$ (w is the winding number of φ). We then define

$$\Gamma(e^{if}) := e^{i\bar{\alpha}Q/2}R^w e^{i\bar{\alpha}Q/2}\Gamma(e^{i(\alpha^+ + \alpha^-)}). \tag{36}$$

This fixes the phase for all implementors. With that we get

$$\sigma(e^{if_1}, e^{if_2}) = e^{-iS(f_1,f_2)/2}, \tag{37}$$

where we introduced

$$S(f_1, f_2) = s(\alpha_1, \alpha_2) + (w_{f_1}\bar{\alpha}_2 - \bar{\alpha}_1 w_{f_2}). \tag{38}$$

It is worth noting that one can write

$$S(f_1,f_2)=f_1(\frac{L}{2})f_2(-\frac{L}{2})-f_1(-\frac{L}{2})f_2(\frac{L}{2}) \\ +\frac{1}{4\pi}\int_{-L/2}^{L/2}dx(\frac{df_1(x)}{dx}f_2(x)-f_1(x)\frac{df_2(x)}{dx}) \tag{39}$$

which (up to trivial, but nevertheless important, rescaling of variables) is identical to the antisymmetric two cocycle introduced by Segal [Se]. Notice that our choice of phase for the implementors implies that

$$\left\langle \Omega, \Gamma(e^{if})\Omega \right\rangle = 0 \quad \text{if } w \neq 0. \tag{40}$$

We will need the following relation:

$$\Gamma(e^{if_1})\Gamma(e^{if_2})\cdots\Gamma(e^{if_N}) = (\prod_{j<k} e^{-iS(f_j,f_k)/2})\Gamma(e^{if_1}e^{if_2}\cdots e^{if_N}) \tag{41}$$

which follows by induction (here and in the following $\prod_{j<k}$ is short for $\prod_{j=1}^{N}\prod_{k=j+1}^{N}$).

We introduce normal ordering $\overset{\times}{\times} \cdots \overset{\times}{\times}$ as follows. For implementors of loops of winding number zero it is defined as

$$\overset{\times}{\times}\Gamma(e^{i\alpha})\overset{\times}{\times} := e^{iS(\alpha^-,\alpha^+)/2}\Gamma(e^{i\alpha}) \tag{42}$$

with the numerical factor chosen such that $\langle \Omega, \overset{\times}{\times}\Gamma(e^{i\alpha})\overset{\times}{\times}\Omega\rangle = 1$ [cf. Eq. (33)] . We extend this to implementors of general loops,

$$\overset{\times}{\times}\Gamma(e^{if})\overset{\times}{\times} := e^{i\bar{\alpha}Q/2}R^w e^{i\bar{\alpha}Q/2}\overset{\times}{\times}\Gamma(e^{i(\alpha^+ + \alpha^-)})\overset{\times}{\times} \tag{43}$$

and to products of implementors,

$$\overset{\times}{\times}\Gamma(e^{if_1})\Gamma(e^{if_2})\cdots\Gamma(e^{if_N})\overset{\times}{\times} := \overset{\times}{\times}\Gamma(e^{if_1}e^{if_2}\cdots e^{if_N})\overset{\times}{\times}\ . \tag{44}$$

A straightforward computation then implies the following relations:

$$\overset{\times}{\times}\Gamma(e^{if_1})\overset{\times}{\times}\overset{\times}{\times}\Gamma(e^{if_1})\overset{\times}{\times} = e^{-i\tilde{S}(f_1,f_2)/2}\overset{\times}{\times}\Gamma(e^{if_1})\Gamma(e^{if_1})\overset{\times}{\times} \tag{45}$$

with

$$\tilde{S}(f_1,f_2) = w_1\bar{\alpha}_2 - \bar{\alpha}_1 w_2 + 2S(\alpha_1^-,\alpha_2^+) = -\tilde{S}(f_2,f_1)^* \tag{46}$$

which will be useful in the following. Finally,

$$\overset{\times}{\times} d\Gamma(\alpha_1)\cdots d\Gamma(\alpha_m)\Gamma(e^{if})\overset{\times}{\times} \\ := (-i)^m \frac{\partial^m}{\partial a_1\cdots\partial a_m}\overset{\times}{\times} e^{ia_1 d\Gamma(\alpha_1)}\cdots e^{ia_m d\Gamma(\alpha_m)}\Gamma(e^{if})\overset{\times}{\times}\Big|_{a_j=0} \tag{47}$$

and

$$\overset{\times}{\times}AB\overset{\times}{\times} = \overset{\times}{\times}BA\overset{\times}{\times} = \overset{\times}{\times}(\overset{\times}{\times}A\overset{\times}{\times})B\overset{\times}{\times} = \overset{\times}{\times}(\overset{\times}{\times}A\overset{\times}{\times})(\overset{\times}{\times}B\overset{\times}{\times})\overset{\times}{\times} \tag{48}$$

extends the definition of normal ordering to arbitrary products of operators $d\Gamma(\alpha_j)$ and $\Gamma(e^{if_k})$. We note that by Stone's theorem [RS1] the differentiations here are well-defined in the strong sense on the dense domain $\mathcal{D}$ defined in Appendix B.

It is convenient to introduce the operators

$$\hat{\rho}(p) := d\Gamma(\epsilon_p), \quad \epsilon_p(x) = e^{-ipx}, \quad p \in \Lambda^* \tag{49}$$

which allow us to write

$$d\Gamma(\alpha) = \sum_{p\in\Lambda} \hat{\alpha}(p)\hat{\rho}(-p). \tag{50}$$

The $\hat{\rho}(p)$ have a natural interpretation as boson field operators and will be further discussed in Appendix A. The subspace $\mathcal{D}_b$ (finite boson vectors) of $\mathcal{F}$ spanned by vectors of the form

$$\eta_b = \hat{\rho}(-q_1)\cdots\hat{\rho}(-q_n)R^\ell\Omega, \quad q_j > 0, n \in \mathbb{N}_0, \ell \in \mathbb{Z} \tag{51}$$

will be important for us. Note that $\mathcal{D}_b$ is dense in $\mathcal{F}$ (see e.g. [CR]).

4. Vertex Operators

4.1. Boson-fermion correspondence. As a motivation and to introduce notation, we first recall how the boson-fermion correspondence can be derived from the results summarized in the last section [PS, CHu]. In Ref. [Se] a so-called "blip" function was introduced which equals, up to the sign,

$$\frac{e^{i(x-y)2\pi/L} - \lambda}{1 - \lambda e^{i(x-y)2\pi/L}}, \quad 0 < \lambda < 1,$$

which is the exponential of a smoothed out step function. Writing it as $e^{if_{y,\varepsilon}}$ with $\lambda = e^{-2\pi\varepsilon/L}$ one gets

$$f_{y,\varepsilon}(x) = \frac{2\pi}{L}(x-y) + \alpha^+_{y,\varepsilon}(x) + \alpha^-_{y,\varepsilon}(x) \tag{52}$$

with

$$\alpha^\pm_{y,\varepsilon}(x) = \pm i \log(1 - e^{2\pi(\pm i(x-y)-\varepsilon)/L}) = \mp i \sum_{n=1}^\infty \frac{1}{n} e^{\pm 2i\pi n(x-y)/L} e^{-2\pi\varepsilon n/L}. \tag{53}$$

Note that the winding number of $f_{y,\varepsilon}$ equals 1. Since $f_{y,\varepsilon}(x)$ for $\varepsilon \downarrow 0$ converges to $\pi\mathrm{sgn}(x-y)$ we will also use the following suggestive notation,

$$\mathrm{sgn}(x-y;\varepsilon) := \frac{1}{\pi} f_{y,\varepsilon}(x). \tag{54}$$

Later we will also need the function $\delta_{y,\varepsilon}(x) = \partial_x f_{y,\varepsilon}(x)/2\pi$, i.e.

$$\delta_{y,\varepsilon}(x) = \frac{1}{L} + \delta^+_{y,\varepsilon}(x) + \delta^-_{y,\varepsilon}(x) \tag{55}$$

with

$$\delta^\pm_{y,\varepsilon}(x) = \frac{1}{L} \sum_{n>0} e^{\pm 2\pi i(x-y)n/L} e^{-2\pi\varepsilon n/L}. \tag{56}$$

These functions have the following important properties which we summarize as

Lemma 1.

$$
\begin{aligned}
S(\alpha^-_{y,\varepsilon},\alpha^+_{y',\varepsilon'}) &= \alpha^+_{y',\varepsilon+\varepsilon'}(y),\\
S(f_{y,\varepsilon},f_{y',\varepsilon'}) &= \pi\mathrm{sgn}(y-y';\varepsilon+\varepsilon'),\\
S(\delta^\mp_{y,\varepsilon},\alpha^\pm_{y',\varepsilon'}) &= -\delta^\pm_{y',\varepsilon+\varepsilon'}(y).
\end{aligned}
\tag{57}
$$

(The proof of these relations is a straightforward calculation which we skip.)

Then for $\varepsilon > 0$ and integer ν the operators $\phi^\nu_\varepsilon(y) := {}^\times_\times\, \Gamma(e^{i\nu f_{y,\varepsilon}})\, {}^\times_\times = \phi^{-\nu}_\varepsilon(y)^*$ are well-defined, and from Lemma 1 and Eqs. (20) and (37) we conclude

$$
\phi^\nu_\varepsilon(y)\phi^{\nu'}_{\varepsilon'}(y') = e^{-i\pi\nu\nu'\mathrm{sgn}(y-y';\varepsilon+\varepsilon')}\phi^{\nu'}_{\varepsilon'}(y')\phi^\nu_\varepsilon(y). \tag{58}
$$

For odd integers ν,ν' and in the limit $\varepsilon,\varepsilon'\downarrow 0$ these formally become anticommutator relations. This suggests that the $\phi^{\pm 1}_\varepsilon(y)$ in this limit are fermion operators. Indeed one can prove

$$
\hat\psi^*(k) = \lim_{\varepsilon\downarrow 0}\frac{1}{\sqrt{2\pi L}}\int_{-L/2}^{L/2} dy\, \phi^1_\varepsilon(y)e^{iky}, \quad k\in\Lambda_0^* \tag{59}
$$

in the sense of strong convergence on a dense domain (see e.g. [CHu, PS]). This is the central result of the boson-fermion correspondence. We note that this relation also fixes the phase of the unitary operator R.

4.2. Construction of anyons. To construct anyons we have to extend the relations Eq. (58) to non-integer ν,ν'. However, the functions $e^{i\nu f_{y,\varepsilon}(x)}$ are not periodic and thus $\Gamma(e^{i\nu f_{y,\varepsilon}})$ does not exist. To circumvent this problem, we note that $S(f_1,f_2)$ Eq. (38) is invariant under changes $\overline{\alpha}_i\to\overline{\alpha}_i\lambda$ and $w_i\to w_i/\lambda$ with an arbitrary scaling parameter λ. We use this to construct a function $\tilde f_{y,\varepsilon}(x)$ which has the following properties,

$$
\begin{aligned}
&(i) \quad e^{i\nu\tilde f_{y,\varepsilon}(x)} \quad \text{is periodic for all } \nu,\\
&(ii) \quad S(\tilde f_{y,\varepsilon},\tilde f_{y,\varepsilon}) = S(f_{y,\varepsilon},f_{y,\varepsilon}).
\end{aligned}
$$

Since the functions $\nu\tilde f_{y,\varepsilon}(x)$ have winding numbers different from zero, the first requirement can only be fulfilled for ν values which are an integer multiple of some fixed number $\nu_0 > 0$. Then

$$
\tilde f_{y,\varepsilon}(x) = \frac{2\pi}{L\nu_0}x - \frac{2\pi\nu_0}{L}y + \alpha^+_{y,\varepsilon}(x) + \alpha^-_{y,\varepsilon}(x) \tag{60}
$$

has the desired properties. Thus the operators

$$
\phi^\nu_\varepsilon(y) := {}^\times_\times\,\Gamma(e^{i\mu\nu_0\tilde f_{y,\varepsilon}})\,{}^\times_\times = \phi^{-\nu}_\varepsilon(y)^*, \quad \nu := \nu_0\mu, \quad \mu\in\mathbb{Z} \tag{61}
$$

are well-defined for $\varepsilon > 0$, and they obey the exchange relations Eq. (58) but now for all ν,ν' which are integer multiples of ν_0. Thus the theory of loop groups provides a simple and rigorous construction of regularized anyon field operators $\phi^\nu_\varepsilon(x)$.

Remark. To be precise, one should denote the anyon operators defined in Eq. (61) as $\phi^{\nu_0,\mu}_\varepsilon(y)$. Then Eq. (58) would read

$$
\phi^{\nu_0,\mu}_\varepsilon(y)\phi^{\nu_0,\mu'}_{\varepsilon'}(y') = e^{-i\pi\nu_0^2\mu\mu'\mathrm{sgn}(y-y';\varepsilon+\varepsilon')}\phi^{\nu_0,\mu'}_{\varepsilon'}(y')\phi^{\nu_0,\mu}_\varepsilon(y) \quad \mu,\mu'\in\mathbb{Z}.
$$

Making the ν_0-dependence manifest would allow us to obtain slightly more general results. However, it would also lead to a proliferation of indices which is a price we are not willing to pay.

We note that this definition and Eq. (43) imply that the anyon fields are not periodic but the operators

$$\check{\phi}^{\nu}_{\varepsilon}(y) := e^{i\pi\nu\nu_0 Qy/L}\phi^{\nu}_{\varepsilon}(y)e^{i\pi\nu\nu_0 Qy/L} = R^{\nu/\nu_0} \,{}^{\times}_{\times}\, e^{i\nu d\Gamma(\alpha^{+}_{y,\varepsilon}+\alpha^{-}_{y,\varepsilon})} \,{}^{\times}_{\times} \tag{62}$$

are. This suggests that the Fourier modes $\hat{\phi}^{\nu}(p)$ of the anyons fields as defined in Eq. (3) are well-defined operators. In fact:

Proposition 1. *The $\hat{\phi}^{\nu}(p)$ defined in Eq. (3) are operators with $\mathcal{D}_b$ as common, dense, invariant domain. Especially,*

$$\hat{\phi}^{\nu}(0)R^{\ell}\Omega = R^{\ell+\nu/\nu_0}\Omega \quad \forall \ell \in \mathbb{Z}. \tag{63}$$

The proof of this is given in Appendix C. It implies that all vectors $\eta_{\nu,N}(\mathbf{n})$ Eq. (9) are in $\mathcal{D}_b$. This is important due to the following result also proven in Appendix C:

Proposition 2. *For $\eta \in \mathcal{D}_b$,*

$$F^{\nu}_{\eta}(x_1, \dots, x_N) := \lim_{\varepsilon\downarrow 0} \langle \eta, \phi^{\nu}_{\varepsilon}(x_1)\cdots\phi^{\nu}_{\varepsilon}(x_N)\Omega\rangle \tag{64}$$

exists and has the form

$$\begin{aligned} F^{\nu}_{\eta}(x_1, \dots, x_N) = e^{-i\pi\nu^2(x_1+\ldots+x_N)N/L}\,\Delta^{\nu^2}_{N}(x_1, \dots, x_N) \\ \times \mathcal{P}_{\eta}(\nu|e^{-2\pi i x_1/L}, \dots, e^{-2\pi i x_N/L}), \end{aligned} \tag{65}$$

where

$$\Delta^{\nu^2}_{N}(\mathbf{x}) := \lim_{\varepsilon\downarrow 0}\left(\prod_{j=1}^{N}\prod_{k=j+1}^{N} b(x_j - x_k;\varepsilon)\right)^{\nu^2} \tag{66}$$

with b given in Eq. (6) and $\mathcal{P}_{\eta}(\nu|\mathbf{z})$ a symmetric polynomial.[2] *Especially, $F^{\nu}_{\eta}(\mathbf{x}) \in L^2(S^N_L)$.*

Proposition 2 follows from the following explicit formula derived in Appendix C: for η_b Eq. (51),

$$\begin{aligned} F^{\nu}_{\eta_b}(x_1, \cdots, x_N) = \delta_{\ell,N\nu/\nu_0} e^{-i\pi\nu^2(x_1+\ldots+x_N)N/L} \\ \times \prod_{j=1}^{n}\left(\sum_{k=1}^{N}\nu e^{-iq_j x_k}\right)\Delta^{\nu^2}_{N}(x_1, \dots, x_N). \end{aligned} \tag{67}$$

We note that $\Delta^{\nu^2}_{N}(x_1, \dots, x_N)$ equals, up to a constant, to the well-known ground state wave function of the Sutherland model (see e.g. [Su]). This will be further explored in Sect. 6.

Using Eqs. (9) and (3) we now obtain

[2] I.e. a polynomial which is invariant under permutations of the arguments, see [McD].

$$\eta_{\nu,N}(\mathbf{n}) = \lim_{\varepsilon_1,\cdots,\varepsilon_N\downarrow 0}\int_{-L/2}^{L/2} dx_1 e^{ip_1x_1}\cdots\int_{-L/2}^{L/2} dx_N e^{ip_Nx_N}\check{\phi}^{\nu}_{\varepsilon_1}(x_1)\cdots\check{\phi}^{\nu}_{\varepsilon_N}(x_N)\Omega$$
$$= \lim_{\varepsilon_1,\cdots,\varepsilon_N\downarrow 0}\int_{-L/2}^{L/2} dx_1 e^{iP_1x_1}\cdots\int_{-L/2}^{L/2} dx_N e^{iP_Nx_N}\phi^{\nu}_{\varepsilon_N}(x_1)\cdots\phi^{\nu}_{\varepsilon_N}(x_N)\Omega \tag{68}$$

with $p_j = \frac{2\pi}{L}n_j$ and

$$P_j = P_{j,\nu,N}(\mathbf{n}) = \frac{2\pi}{L}\left(n_j + \nu^2(N-j+\frac{1}{2})\right) \quad j = 1,2,\ldots N. \tag{69}$$

(To derive this formula we used repeatedly $e^{icQ}R^w\Omega = e^{icw}R^w\Omega$ for $c \in \mathbb{R}$ and $w \in \mathbb{Z}$.) These P_j can be interpreted as anyon momenta, and they will play an important role in Sect. 6. It is interesting to note how the momentum shifts $\propto \nu^2$ appear in our formalism: they are due to the factors $e^{-\pi\nu\nu_0 Qx/L}$ in Eq. (3) which are necessary to make the anyon operators periodic.

We finally formulate a *highest weight condition* for the Fourier modes of the anyon field operators which is analogous to Eq. (136) in Appendix A and will also play an important role in Sect. 6.

Proposition 3. *The vector* $\eta_{\nu,N}(\mathbf{n})$ *Eq. (9) is non-zero only if the following conditions are fulfilled,*

$$n_1 + n_2 + \ldots + n_N \geq 0, \tag{70}$$
$$n_\ell + \sum_{j=\ell+1}^{N} 2^{j-1-\ell}n_j \geq 0 \quad \textit{for } \ell = 1,2,\ldots N. \tag{71}$$

Again we defer the proof to Appendix C.

4.3. Anyon correlation functions. The results of the last two subsections enable us to complete one of our main aims namely to compute all anyon correlations functions. First Eqs. (42), (44), and (57) imply

$$\phi^{\nu_1}_{\varepsilon_1}(x_1)\cdots\phi^{\nu_N}_{\varepsilon_N}(x_N) = \mathcal{J}^{\nu_1,\cdots,\nu_N}_{\varepsilon_1,\cdots,\varepsilon_N}(x_1,\ldots,x_N)\, {}^{\times}_{\times}\phi^{\nu_1}_{\varepsilon_1}(x_1)\cdots\phi^{\nu_N}_{\varepsilon_N}(x_N){}^{\times}_{\times} \tag{72}$$

where

$$\mathcal{J}^{\nu_1,\cdots,\nu_N}_{\varepsilon_1,\cdots,\varepsilon_N}(x_1,\ldots,x_N) = \prod_{j<k} b(x_j-x_k;\varepsilon_j+\varepsilon_k)^{\nu_j\nu_k} \tag{73}$$

and the function $b(r,\varepsilon)$ is defined in Eq. (6). Note that our definition of normal ordering implies

$$\left\langle \Omega, R^{w_1}\, {}^{\times}_{\times}\phi^{\nu_1}_{\varepsilon_1}(x_1)\cdots\phi^{\nu_N}_{\varepsilon_N}(x_N){}^{\times}_{\times}\, R^{w_2}\Omega\right\rangle$$
$$= \delta_{w_1+w_2+(\nu_1+\ldots+\nu_N)/\nu_0,0}\, e^{i\pi(w_1-w_2)\nu_0(\nu_1x_1+\ldots+\nu_Nx_N)/L}. \tag{74}$$

Now using Eqs. (33), (40) we obtain Eqs. (4)–(5). Our main interest is in the functions defined in Eq. (8) and which can be written as

$$f_{\nu,N}(\mathbf{n}|\mathbf{x}) = F^{\nu}_{\eta_{\nu,N}(\mathbf{n})}(\mathbf{x}). \tag{75}$$

By a simple computation,

$$f_{\nu,N}(\mathbf{n}|\mathbf{x}) = \lim_{\varepsilon\downarrow 0} \lim_{\varepsilon_1,\ldots,\varepsilon_N\downarrow 0} \int_{-L/2}^{L/2} dy_1 e^{-iP_1 y_1} \cdots \int_{-L/2}^{L/2} dy_N e^{-iP_N y_N}$$
$$\times \left\langle \Omega, \phi_{\varepsilon_N}^{-\nu}(y_N) \cdots \phi_{\varepsilon_1}^{-\nu}(y_1) \phi_\varepsilon^\nu(x_1) \cdots \phi_\varepsilon^\nu(x_N)\Omega \right\rangle$$
$$= e^{-i\pi\nu^2(x_1+\ldots+x_N)N/L} \Delta_N^{\nu^2}(x_1,\ldots,x_N) \lim_{\varepsilon\downarrow 0} \int_{-L/2}^{L/2} dy_1 e^{-ip_1 y_1} \cdots \int_{-L/2}^{L/2} dy_N e^{-ip_N y_N}$$
$$\times \prod_{j>j'} \check{b}(y_j - y_{j'}; \varepsilon_j + \varepsilon_{j'})^{\nu^2} \prod_{j,\ell} \check{b}(y_j - x_\ell; 2\varepsilon)^{-\nu^2},$$

where $p_j = \frac{2\pi}{L} n_j$ and $\check{b}(x,\varepsilon) := \left(1 - e^{-\frac{2\pi}{L}\varepsilon} e^{i\frac{2\pi}{L}x}\right)$. Comparing with Eq. (65) we see that

$$\mathcal{P}_{\eta_{\nu,N}(\mathbf{n})}(z_1,\ldots,z_N) = \lim_{\varepsilon\downarrow 0} \int_{-L/2}^{L/2} dy_1 e^{-i\frac{2\pi}{L}n_1 y_1} \cdots \int_{-L/2}^{L/2} dy_N e^{-i\frac{2\pi}{L}n_N y_N}$$
$$\times \prod_{j>j'} \left(1 - e^{-\frac{2\pi}{L}\varepsilon} e^{i\frac{2\pi}{L}(y_j - y_{j'})}\right)^{\nu^2} \prod_{j,\ell} \left(1 - e^{-\frac{2\pi}{L}\varepsilon} e^{i\frac{2\pi}{L}y_j} z_\ell\right)^{-\nu^2}.$$

We now can expand the integrand in a Taylor series in the exponentials and then perform the y_j-integrations. The final result is

$$\mathcal{P}_{\eta_{\nu,N}(\mathbf{n})}(z_1,\ldots,z_N) = L^N {\sum}' \prod_{j=1}^{N} \prod_{j'=1}^{j-1} \prod_{\ell=1}^{N} \binom{\nu^2}{\mu_{jj'}} \binom{-\nu^2}{m_{j\ell}} (-1)^{\mu_{jj'}} (-z_\ell)^{m_{j\ell}}, \quad (76)$$

where $\binom{\pm\nu^2}{n}$ are the binomial coefficients as usual and ${\sum}'$ here means summation over all $\mu_{jj'}, m_{j\ell} \in \mathbb{N}_0$ such that

$$\sum_{j'=1}^{j-1} \mu_{jj'} - \sum_{j'=j+1}^{N} \mu_{j'j} + \sum_{\ell=1}^{N} m_{j\ell} = n_j \quad \text{for } j = 1, 2 \ldots N\,. \quad (77)$$

4.4. The braid group. The braid group will not play a role in our deliberations, however we mention one observation for completeness. We define operators on the N-anyon subspace as follows. On a vector

$$\phi_\varepsilon^\nu(x_1) \cdots \phi_\varepsilon^\nu(x_N)\Omega$$

define, for $i \in \{1, 2, ..., N-1\}$, σ_i to be the operator which interchanges the i^{th} and $(i+1)^{th}$ arguments and multiplies by the phase:

$$e^{-i\pi\nu^2 \mathrm{sgn}(x_i - x_{i+1};\varepsilon)/2}.$$

An easy calculation reveals that the braid relations hold:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1,$$
$$\sigma_i^2 = 1,$$
$$\sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i, \quad |i-j| = 1.$$

So we have a braid group action on each N-anyon subspace.

5. W-Charges

5.1. Motivation. There are self-adjoint operators W^s on $\mathcal{F}$ obeying

$$[W^s, \hat{\psi}^*(k)] = k^{s-1}\hat{\psi}^*(k) \quad \forall k \in \Lambda_0^*, \quad W^s\Omega = 0 \qquad (s \in \mathbb{N}). \tag{78}$$

If we introduce an operator valued distribution $\psi^*(x)$ such that

$$\hat{\psi}^*(k) = \psi^*(e_k) = \int_{S_L} dx\, e_k(x)\psi^*(x),$$

the commutator relations in Eq. (78) are (formally[3]) equivalent to

$$[W^s, \psi^*(x)] = i^{s-1}\frac{\partial^{s-1}}{\partial x^{s-1}}\psi^*(x). \tag{79}$$

These operators W^s can be represented in terms of the operators $\hat{\rho}(p)$ Eq. (49),

$$\begin{aligned} W^1 &= \hat{\rho}(0), \\ W^2 &= \frac{\pi}{L}\sum_{p\in\Lambda^*} {}^\times_\times \hat{\rho}(p)\hat{\rho}(-p) {}^\times_\times, \\ W^3 &= \frac{4\pi^2}{3L^2}\sum_{p_1,p_2\in\Lambda^*} {}^\times_\times \hat{\rho}(p_1)\hat{\rho}(p_2)\hat{\rho}(-p_1-p_2) {}^\times_\times - \frac{\pi^2}{3L^2}\hat{\rho}(0), \\ &\text{etc.} \end{aligned} \tag{80}$$

These formulas are known in the physics literature (see e.g. [B]). We shall construct operators which obey similar relations with the anyon field operators $\phi_\varepsilon^\nu(x)$. To explain our method, we will first present a construction of operators W^s obeying Eq. (78) for all $s \in \mathbb{N}$. We then show how to partly extend this to anyons. The extension is essentially trivial for $s = 1, 2$. The first non-trivial case is $s = 3$. We propose a natural generalization of W^3 and show that it corresponds to a "second quantization" of the CS Hamiltonian Eq. (7), as described in the Introduction.

To simplify our notation, we set $\nu_0 = \nu > 0$ in the rest of the paper.

5.2. W-charges for fermions. We define

$$\mathcal{W}_\varepsilon^\nu(y; a) := N^\nu(a)\left({}^\times_\times e^{i\nu d\Gamma(\tilde{f}_{y+a,\varepsilon} - \tilde{f}_{y,\varepsilon})} {}^\times_\times - I\right), \tag{81}$$

with functions $\tilde{f}_{y,\varepsilon}$ given by Eqs. (60), (53) and the normalization constant

$$N^\nu(a) = \frac{i}{2L\nu^2\cos^{\nu^2}(\frac{\pi}{L}a)\tan(\frac{\pi}{L}a)}. \tag{82}$$

In this section we are mainly interested in the fermion case where $\nu = 1$, but in our discussion on anyons later we will need these formulas for general non-zero $\nu \in \mathbb{R}$.

We claim that

$$\mathcal{W}^\nu(a) := \lim_{\varepsilon\downarrow 0}\int_{-L/2}^{L/2} dy\, \mathcal{W}_\varepsilon^\nu(y; a) = \sum_{s=1}^\infty \frac{(-ia)^{s-1}\nu^{s-2}}{(s-1)!}W^{\nu,s} \tag{83}$$

defines an operator valued generating function for operators $W^{\nu,s}$, $s \in \mathbb{N}$. To be more precise:

[3] Our results below will actually give a precise mathematical meaning to this

Lemma 2. *For all $a \in \mathbb{R}$ and non-zero $\nu \in \mathbb{R}$, the operators $\mathcal{W}^\nu(a)$ Eqs. (81)–(83) are well-defined on $\mathcal{D}_b$ and leave $\mathcal{D}_b$ invariant. Especially,*

$$\mathcal{W}^\nu(a)\Omega = 0. \tag{84}$$

Moreover, Eq. (83) defines a family of operators $W^{\nu,s}$, $s \in \mathbb{N}$, which have $\mathcal{D}_b$ as a common, dense invariant domain of definition.

The proof of this result is in Appendix D. We now show how to compute these operators $W^{\nu,s}$ explicitly. We define

$$\tilde{\delta}_{y,\varepsilon}(x) := -\frac{1}{2\pi}\partial_y \tilde{f}_{y,\varepsilon}(x) = \delta_{y,\varepsilon}(x) + \frac{(1-\nu)}{L} \tag{85}$$

where $\partial_y = \frac{\partial}{\partial y}$, and[4]

$$\tilde{\rho}_\varepsilon(y) := d\Gamma(\tilde{\delta}_{y,\varepsilon}) = \rho_\varepsilon(y) + \frac{(1-\nu)}{L}Q. \tag{86}$$

With that we obtain

$$d\Gamma(\tilde{f}_{y+a,\varepsilon} - \tilde{f}_{y,\varepsilon}) = -2\pi \sum_{k=1}^{\infty} \frac{a^s}{s!}\partial_y^{s-1}\tilde{\rho}_\varepsilon(y),$$

and one can expand $\mathcal{W}^\nu_\varepsilon(y;a)$ Eq. (83) in a formal power series in a. A straightforward computation then gives

$$\begin{aligned}
W^{\nu,1} &= \int_{-L/2}^{L/2} dy \, {}^\times_\times \tilde{\rho}_\varepsilon(y) {}^\times_\times \Big|_{\varepsilon\downarrow 0}, \\
W^{\nu,2} &= \pi \int_{-L/2}^{L/2} dy \, {}^\times_\times \tilde{\rho}_\varepsilon(y)^2 {}^\times_\times \Big|_{\varepsilon\downarrow 0}, \\
W^{\nu,3} &= \frac{4\pi^2}{3} \int_{-L/2}^{L/2} dy \, {}^\times_\times \tilde{\rho}_\varepsilon(y)^3 {}^\times_\times \Big|_{\varepsilon\downarrow 0} + \frac{\pi^2}{3L^2\nu^2}(2-3\nu^2)W^{\nu,1}, \\
&\text{etc.}
\end{aligned} \tag{87}$$

(this list can be easily extended with the help of a symbolic programming language like MAPLE). Note that for $\nu = 1$, these are identical to the operators in Eq. (80), $W^{1,s} = W^s$ for $s = 1, 2, 3$. Later we will also need the following formulas which are obtained by simple computations from the definitions above,

$$\begin{aligned}
W^{\nu,1} &= (2-\nu)Q, \\
W^{\nu,2} &= W^2 + \frac{\pi}{L}(1-\nu)(1-3\nu)Q^2, \\
W^{\nu,3} &= W^3 + 4\frac{\pi}{L}(1-\nu)QW^2 + \frac{4}{3}\left(\frac{\pi}{L}\right)^2(1-\nu)^2(4-\nu)Q^3 \\
&\quad -\frac{2}{3}\left(\frac{\pi}{L}\right)^2(1-\nu)(1-\nu-3\nu^2)Q, \\
&\quad \text{etc.}
\end{aligned} \tag{88}$$

The result described in the last subsection can now be stated as follows.

[4] Note that $\hat{\rho}(p) = \lim_{\varepsilon\downarrow 0}\int_{-L/2}^{L/2} dx\, \rho_\varepsilon(x)e^{-ipx}$, which motivates our notation.

Theorem 1. *The operators $W^{1,s}$ obey the relations Eq. (78), i.e. $W^{1,s} = W^s$ for all $s \in \mathbb{N}$.*

Proof. We recall Eq. (84) for $\nu = 1$. Here we will show that

$$[\mathcal{W}^1(a), \hat{\psi}^*(k)] = e^{-ika}\hat{\psi}^*(k). \tag{89}$$

These two relations prove the result, as can be seen by an expansion in a formal power series in a and using Eq. (83).

To prove Eq. (89) we use the boson-fermion correspondence Eq. (59). We thus compute the commutator of $\mathcal{W}^1_{\varepsilon'}(y)$ with $\phi^1_\varepsilon(x) = \Gamma(e^{if_{x,\varepsilon}})$. With Eqs. (45), (46) and (57) we obtain

$$[\mathcal{W}^1_{\varepsilon'}(y;a), \phi^1_\varepsilon(x)] = (\cdots) \, {}^\times_\times \Gamma(e^{i[f_{x,\varepsilon}+f_{y+a,\varepsilon'}-f_{y,\varepsilon'}]}) {}^\times_\times$$

with

$$(\cdots) := N^1(a)\left(\frac{\sin\frac{\pi}{L}(y+a-x+i\tilde{\varepsilon})}{\sin\frac{\pi}{L}(y-x+i\tilde{\varepsilon})} - c.c.\right) = \frac{i}{2L}\left(\cot\frac{\pi}{L}(y-x+i\tilde{\varepsilon}) - c.c.\right),$$

where $\tilde{\varepsilon} = \varepsilon + \varepsilon'$ and $c.c.$ means the same term complex conjugated. We now use that

$$\pm\frac{i}{2L}\cot\frac{\pi}{L}(y-x\pm i\tilde{\varepsilon}) = \frac{1}{2L} + \delta^\pm_{x,\tilde{\varepsilon}}(y) \tag{90}$$

which is easily seen by expanding the l.h.s as a Taylor series in $e^{\pm i(y-x)2\pi/L}e^{-\varepsilon 2\pi/L}$. Thus $(\cdots) = \delta_{x,\tilde{\varepsilon}}(y)$ independent of a (!), and we obtain

$$[\mathcal{W}^1_{\varepsilon'}(y;a), \phi^1_\varepsilon(x)] = \delta_{x,\varepsilon+\varepsilon'}(y) \, {}^\times_\times \Gamma(e^{i[f_{x,\varepsilon}+f_{y+a,\varepsilon'}-f_{y,\varepsilon'}]}) {}^\times_\times .$$

Using Eqs. (83) and (59) we thus obtain for the l.h.s. of Eq. (89),

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\sqrt{2\pi L}}\int_{-L/2}^{L/2} dx\, e^{ikx} \lim_{\varepsilon'\downarrow 0}\int_{-L/2}^{L/2} dy\, \delta_{x,\varepsilon+\varepsilon'}(y) \, {}^\times_\times \Gamma(e^{i[f_{x,\varepsilon}+f_{y+a,\varepsilon'}-f_{y,\varepsilon'}]}) {}^\times_\times$$
$$= \lim_{\varepsilon\downarrow 0}\frac{1}{\sqrt{2\pi L}}\int_{-L/2}^{L/2} dx\, e^{ikx} \, {}^\times_\times \Gamma(e^{if_{x+a,\varepsilon}}) {}^\times_\times$$

in the sense of strong convergence on a dense domain. Recalling $\Gamma(e^{if_{x,\varepsilon}}) = \phi^1_\varepsilon(x)$ and using Eq. (59) again we obtain the r.h.s. of Eq. (89). □

We finally discuss a technical point which will be important in the next section: Our proof above shows that

$$[\mathcal{W}^1(a), \phi^1_\varepsilon(x)] \simeq \phi^1_\varepsilon(x+a),$$

where "$\simeq$" means equality after smearing with appropriate test functions and taking the strong limit $\varepsilon \downarrow 0$ on an appropriate dense domain. It will be useful to characterize "$\simeq$" more explicitly as follows. Using Eq. (61) we define

$$\tilde{\phi}^\nu_\varepsilon(x;a) := \lim_{\varepsilon'\downarrow 0}\int_{-L/2}^{L/2} dy\, \delta_{x,\varepsilon+\varepsilon'}(y) \, {}^\times_\times \Gamma(e^{i\nu[\tilde{f}_{x,\varepsilon}+\tilde{f}_{y+a,\varepsilon'}-\tilde{f}_{y,\varepsilon'}]}) {}^\times_\times . \tag{91}$$

Then $\tilde{\phi}^\nu_\varepsilon(x;a) \simeq \phi^\nu_\varepsilon(x+a)$. We now define

$$\frac{\partial_\varepsilon^{s-1}}{\partial_\varepsilon x^{s-1}}\phi^\nu_\varepsilon(x) := \left.\frac{\partial^{s-1}}{\partial x^{s-1}}\tilde{\phi}^\nu_\varepsilon(x;a)\right|_{a=0} \tag{92}$$

for $s = 1, 2, \ldots$, which we regard as ε-deformed differentiations. We specify the relation between these and the ordinary differentiations in the following

Lemma 3.

$$\frac{\partial_\varepsilon^{s-1}}{\partial_\varepsilon x^{s-1}}\phi_\varepsilon^\nu(x) = \frac{\partial^{s-1}}{\partial x^{s-1}}\phi_\varepsilon^\nu(x) + \varepsilon \,{}^\times_\times c_\varepsilon^{s,\nu}(x)\phi_\varepsilon^\nu(x)\,{}^\times_\times, \tag{93}$$

where $c_\varepsilon^{s,\nu}(x)$ is a well-defined operator-valued distribution for $\varepsilon \downarrow 0$. Especially,[5]

$$c_\varepsilon^{1,\nu}(x) = c_\varepsilon^{2,\nu}(x) = 0,$$

$$c_\varepsilon^{3,\nu}(x) = \frac{(i\nu)^2}{L^2}\sum_{p_1,p_2\in\Lambda^*}\hat\rho(p_1)\hat\rho(p_2)\,e^{i(p_1+p_2)x}\frac{1}{\varepsilon}\left(e^{-\varepsilon(|p_1+p_2|)} - e^{-\varepsilon|p_1|-\varepsilon|p_2|}\right). \tag{94}$$

(The proof is a straightforward computation which we skip.)

5.3. W-charges for anyons. The considerations of the preceding section may be extended to cover the case of anyons i.e. ν an arbitrary non-zero real number. Using an argument similar to that in the proof of Theorem 1, we compute

$$[\mathcal{W}_{\varepsilon'}^\nu(y;a), \phi_\varepsilon^\nu(x)] = (\cdots)\,{}^\times_\times\Gamma(e^{i\nu[\tilde f_{x,\varepsilon}+\tilde f_{y+a,\varepsilon'}-\tilde f_{y,\varepsilon'}]})\,{}^\times_\times$$

with $(\cdots)$ equal to

$$N^\nu(a)\left[\left(\frac{\sin\frac{\pi}{L}(y+a-x+i\tilde\varepsilon)}{\sin\frac{\pi}{L}(y-x+i\tilde\varepsilon)}\right)^{\nu^2} - c.c.\right]$$

$$= N^\nu(a)\cos^{\nu^2}(\frac{\pi}{L}a)\left(1+\tanh(\frac{\pi}{L}a)\cot\frac{\pi}{L}(y-x+i\tilde\varepsilon)\right)^{\nu^2} + c.c.$$

$$= \delta_{x,\tilde\varepsilon}(y) - \frac{1}{2}(\nu^2-1)a\partial_y\delta_{x,\tilde\varepsilon}(y) + \mathcal{O}(a^2), \tag{95}$$

where $\tilde\varepsilon = \varepsilon + \varepsilon'$ (in the last line we Taylor expanded in a and used $\cot^2(z) = -1 - d\cot(z)/dz$ and Eq. (90)). Integrating this in y, performing a partial integration, and using Eq. (91) we thus obtain

$$[\mathcal{W}^\nu(a), \phi_\varepsilon^\nu(x)] = \tilde\phi_\varepsilon^\nu(x;a) + i\pi\nu(\nu^2-1)a\,{}^\times_\times[\tilde\rho_\varepsilon(x+a)-\tilde\rho_\varepsilon(x)]\tilde\phi_\varepsilon^\nu(x;a)\,{}^\times_\times + \mathcal{O}(a^3).$$

Comparing now equal powers of a on both sides of the last equation using Eqs. (91)–(94) we see that the generalization of Theorem 1 to anyons holds true only for $s = 1, 2$,

$$[W^{\nu,s}, \phi_\varepsilon^\nu(x)] = \nu^{2-s}i^{s-1}\frac{\partial^{s-1}}{\partial x^{s-1}}\phi_\varepsilon^\nu(x) \qquad s = 1,2, \tag{96}$$

but for $s > 2$ we get correction terms, e.g.

$$[W^{\nu,3}, \phi_\varepsilon^\nu(x)] = \frac{i^2}{\nu}\frac{\partial_\varepsilon^2}{\partial_\varepsilon x^2}\phi_\varepsilon^\nu(x) + 2\pi i(\nu^2-1)\,{}^\times_\times\tilde\rho_\varepsilon(x)'\phi_\varepsilon^\nu(x)\,{}^\times_\times, \tag{97}$$

where $\tilde\rho_\varepsilon(x)' := \partial_x\tilde\rho_\varepsilon(x)$. We define

$$\mathcal{H}^{\nu,1} := \frac{1}{\nu}W^{\nu,1}, \quad \mathcal{H}^{\nu,2} := W^{\nu,2} \tag{98}$$

[5] We will only need this for $s = 1, 2, 3$ and thus do not specify the $c_\varepsilon^{s,\nu}(x)$ for $s > 3$.

which according to Eq. (96) are the anyon W-charges for $s = 1, 2$.

In the following we only consider the first non-trivial case $s = 3$. To proceed, it is crucial to observe that the correction term in Eq. (97) can be partly canceled using the following operator,

$$\mathcal{C} = -\pi i \int_{-L/2}^{L/2} dy \; {}^{\times}_{\times} [\rho^+_\varepsilon(y) - \rho^-_\varepsilon(y)]\partial_y[\rho^+_\varepsilon(y) + \rho^-_\varepsilon(y)] {}^{\times}_{\times} \Big|_{\varepsilon\downarrow 0}$$
$$= -\frac{2\pi}{L} \sum_{p>0} {}^{\times}_{\times} p\hat{\rho}(p)\hat{\rho}(-p) {}^{\times}_{\times}, \tag{99}$$

where

$$\rho^{\pm}_{y,\varepsilon} := d\Gamma(\delta^{\pm}_{y,\varepsilon}). \tag{100}$$

This operator obeys the remarkable relations,

$$\mathcal{C}\phi^\nu_\varepsilon(x) + \phi^\nu_\varepsilon(x)\mathcal{C} = 2\pi i\nu \; {}^{\times}_{\times} \tilde{\rho}_\varepsilon(x)'\phi^\nu_\varepsilon(x) {}^{\times}_{\times} + 2 \; {}^{\times}_{\times} \mathcal{C}\phi^\nu_\varepsilon(x) {}^{\times}_{\times} \; . \tag{101}$$

The proof of this, which we now outline, is by a computation similar to the one leading to Eq. (97). We consider the operator

$$\mathcal{V}_\varepsilon(y; a, b) := {}^{\times}_{\times} e^{-iad\Gamma(i\delta^+_{y,\varepsilon} - i\delta^-_{y,\varepsilon}) + ibd\Gamma(\partial_y\delta^+_{y,\varepsilon} + \partial_y\delta^-_{y,\varepsilon})} {}^{\times}_{\times} \tag{102}$$

and observe that

$$\mathcal{C} = -\pi \lim_{\varepsilon\downarrow 0} \int_{-L/2}^{L/2} dx \, \frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathcal{V}_\varepsilon(y; a, b)\Big|_{a=b=0} . \tag{103}$$

Using Eqs. (45), (46) and (57) one then computes

$$\mathcal{V}_{\varepsilon'}(y; a, b)\phi^\nu_\varepsilon(x) + \phi^\nu_\varepsilon(x)\mathcal{V}_{\varepsilon'}(y; a, b)$$

which by a Taylor expansion in a and b and integrating in y gives Eq. (101). (For details see Appendix D, Proof of Lemma 4.)

We also note that Eq. (138) implies

$$\mathcal{C}R^\ell\Omega = 0 \quad \forall \ell \in \mathbb{Z}. \tag{104}$$

Thus the operator

$$\mathcal{H}^{\nu,3} = \nu W^{\nu,3} + (1-\nu^2)\mathcal{C} \tag{105}$$

obeys the relation

$$[\mathcal{H}^{\nu,3}, \phi^\nu_\varepsilon(x)] = i^2 \frac{\partial^2_\varepsilon}{\partial_\varepsilon x^2}\phi^\nu_\varepsilon(x) + 2(1-\nu^2)({}^\times_\times \mathcal{C}\phi^\nu_\varepsilon(x) {}^\times_\times - \phi^\nu_\varepsilon(x)\mathcal{C}).$$

Again there are correction terms, however, in contrast to the one in Eq. (97) it vanishes when applied to vectors $R^w\Omega$! We obtain

$$[\mathcal{H}^{\nu,3}, \phi^\nu_\varepsilon(x)]R^w\Omega = i^2\frac{\partial^2}{\partial x^2}\phi^\nu_\varepsilon(x)R^w\Omega \tag{106}$$

(we used Lemma 3 and ${}^\times_\times c^{s,\nu}_\varepsilon(x)\phi^\nu_\varepsilon(x) {}^\times_\times R^w\Omega = 0$). This seems to be the best we can do to generalize the relation Eq. (79) for $s = 3$ to the anyon case.

To fully appreciate this operator $\mathcal{H}^{\nu,3}$ one has to extend the computation above to a product of multiple anyon operators. We thus obtain our main result:

Theorem 2. *The operator $\mathcal{H}^{\nu,3}$ obeys the following relations,*

$$[\mathcal{H}^{\nu,3}, \phi^\nu_{\varepsilon_1}(x_1)\cdots\phi^\nu_{\varepsilon_N}(x_N)]R^w\Omega = H_{N,\nu^2,\boldsymbol{\varepsilon}}\phi^\nu_{\varepsilon_1}(x_1)\cdots\phi^\nu_{\varepsilon_N}(x_N)R^w\Omega \tag{107}$$

for all integers w, where

$$H_{N,\nu^2,\boldsymbol{\varepsilon}} = -\sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} + \sum_{\substack{k,\ell=1\\k\neq\ell}}^N \frac{(\frac{\pi}{L})^2\nu^2(\nu^2-1)}{\sin^2\frac{\pi}{L}(x_k - x_\ell - i\,\mathrm{sgn}(k-\ell)(\varepsilon_k+\varepsilon_\ell))} + C_{N,\nu^2,\boldsymbol{\varepsilon}}(\mathbf{x}) \tag{108}$$

is a regularized version of the CS Hamiltonian Eq. (7), i.e. the function $C_{N,\nu^2,\boldsymbol{\varepsilon}}(\mathbf{x})$[6] *is non-singular and vanishes uniformly as $\varepsilon_j \downarrow 0$ for all $j = 1, 2. \dots N$.*

The proof of this theorem is a straightforward but tedious extension of the computation leading to Eq. (106) (which is the special case $N = 1$), and the interested reader can find it in Appendix D.

6. The Calogero–Sutherland Hamiltonian and its Eigenfunctions

We are now ready to show how the results of the last section provide the means to construct eigenfunctions and the corresponding eigenvalues of the CS Hamiltonian Eq. (7).

6.1. Eigenfunctions from anyon correlation functions. We claim that Theorem 2 essentially relates these eigenfunctions of the Sutherland Hamiltonian H_{N,ν^2} Eq. (7), to the eigenvectors of the operator $\mathcal{H}^{\nu,3}$. In fact the key step is just to observe the following elementary corollary of Theorem 2.

[6] The interested reader can find the definition of this function in Eq. (156) below.

Proposition 4. *Let $\eta \in \mathcal{D}_b$. Then*

$$\lim_{\varepsilon\downarrow 0} < \eta, \mathcal{H}^{\nu,3}\phi^\nu_\varepsilon(x_1)\cdots\phi^\nu_\varepsilon(x_N)\Omega >= H_{N,\nu^2}F^\nu_\eta(x_1,\ldots,x_N), \tag{109}$$

where F^ν_η is defined in Eq. (64) Especially, if η is an eigenvector of $\mathcal{H}^{\nu,3}$ with the eigenvalue $\mathcal{E}$, then F^ν_η is an eigenfunction of H_{N,ν^2} with the same eigenvalue $\mathcal{E}$.

The immediate next question is to ask if our method constructs all eigenvectors of (7). We answer this in two steps. We first state and prove another consequence of Theorem 2.

Proposition 5. *The vectors $\eta_{\nu,N}(\mathbf{n})$ defined in Eq. (9) are in $\mathcal{D}_b$, and they obey*

$$\mathcal{H}^{\nu,3}\eta_{\nu,N}(\mathbf{n}) = \mathcal{E}_{\nu,N}(\mathbf{n})\eta_{\nu,N}(\mathbf{n}) + \gamma\sum_{j<\ell}\sum_{n=1}^{\infty} n\eta_{\nu,N}(\mathbf{n}+n[\mathbf{e}_j-\mathbf{e}_\ell]) \tag{110}$$

with

$$\mathcal{E}_{\nu,N}(\mathbf{n}) = \sum_{j=1}^{N} P_j^2, \tag{111}$$

P_j defined in Eq. (69),

$$\gamma := 2\nu^2(\nu^2-1)\left(\frac{2\pi}{L}\right)^2, \tag{112}$$

and $\mathbf{e}_1 = (1,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,\ldots,0)$, $\ldots$, $\mathbf{e}_N = (0,0,\ldots,1)$.

Proof. We use Eqs. (9), (3) and Theorem 2 to write

$$\mathcal{H}^{\nu,3}\eta_{\nu,N}(\mathbf{n}) = (\cdot)_1 + (\cdot)_2 + (\cdot)_3$$

with

$$(\cdot)_1 = \sum_{j=1}^{N}\lim_{\varepsilon_1,\ldots,\varepsilon_N\downarrow 0}\int_{-L/2}^{L/2}dx_1 e^{iP_1x_1}\cdots\int_{-L/2}^{L/2}dx_N e^{iP_Nx_N} \times\left(-\frac{\partial^2}{\partial x_j^2}\right)\phi^\nu_{\varepsilon_1}(x_1)\cdots\phi^\nu_{\varepsilon_N}(x_N)\Omega,$$

$$(\cdot)_2 = \lim_{\varepsilon_1,\ldots,\varepsilon_N\downarrow 0}\int_{-L/2}^{L/2}dx_1 e^{ip_1x_1}\cdots\int_{-L/2}^{L/2}dx_N e^{ip_Nx_N} \times C_{N,\nu^2,\boldsymbol{\varepsilon}}(x_1,\ldots,x_N)\check{\phi}^\nu_{\varepsilon_1}(x_1)\cdots\check{\phi}^\nu_{\varepsilon_N}(x_N)\Omega$$

and

$$(\cdot)_3 = \sum_{j<\ell}\gamma\lim_{\varepsilon_1,\ldots,\varepsilon_N\downarrow 0}\int_{-L/2}^{L/2}dx_1 e^{ip_1x_1}\cdots\int_{-L/2}^{L/2}dx_N e^{ip_Nx_N} \times\mathcal{S}(x_j-x_\ell;\varepsilon_k+\varepsilon_\ell)\check{\phi}^\nu_{\varepsilon_1}(x_1)\cdots\check{\phi}^\nu_{\varepsilon_N}(x_N)\Omega,$$

where γ is defined in Eq. (112) and