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Fourier-Malliavin Volatility Estimation

Theory and Practice

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To our families

Preface

The concept of *volatility* refers to any phenomenon presenting features of instability, unpredictability, and a likeliness to change frequently, often without apparent or cogent reason; in a word, a phenomenon that exhibits random variations. Therefore, it is an essential element of almost all branches of science, and the measurement of its impact and effects is of paramount importance. This book mainly focuses on the measurement of the statistical parameter which Bachelier (1900) called "nervosité" (the coefficient of nervousness) of a market price and which nowadays is referred as variance or volatility in the context of financial applications. Nevertheless, many of the methods and results presented here could be applied to other disciplines (from turbulence to chemistry, from physics to computer science and even medicine).

Ideally, we start from the book chapter "Volatility Estimation by Fourier Expansion" by Malliavin and Thalmaier (2006) and follow the rapid development of the Fourier-Malliavin estimation theory over the last decade. The purpose of this book is to give a picture of the state of the art concerning this theory and to suggest new directions for its application in the study of financial markets. We aim to give the interested reader a clear, comprehensive, and self-contained book on the use of the Fourier-Malliavin technique for volatility estimation, providing the theoretical and numerical tools needed to understand and apply the methodology to real cases. Specifically, readers are given examples and instruments to implement this methodology in various financial settings, and some new applications to real data are proposed. Detailed bibliographic references are pointed out to permit a study in depth. This book will appeal to the financial econometrics and quantitative finance community and, in particular, to PhD students, researchers, and practitioners in these fields.

Chapter 1 briefly introduces the main elements, namely, various concepts of volatility, the peculiar characteristics of market (high-frequency) data, and the Fourier analysis for financial time series. In Chapter 2, the reader is introduced to the basic idea underlying the Fourier-Malliavin method, and some intuitions on the method are anticipated. Chapter 3 mainly focuses on estimating integrated volatility and cross-volatility on a fixed time horizon, e.g. a day, while in Chapter 4, the Fourier estimation of instantaneous volatility is studied. In Chapter 5, the efficiency of the estimation method is analyzed when the observed asset prices are contaminated by market microstructure noise effects, as it happens when high-frequency data are employed. Chapter 6 gives some examples of the potential of the Fourier method to deal with the real-time use of the volatility estimates. The essentials of the mathematical background are presented in Appendix A, which enables the non-expert reader to follow the theory presented in the book. Furthermore, Appendix B provides a collection of MATLAB® codes useful for reproducing the numerical results contained in the book.

Acknowledgments

This book could not have existed without Professor Malliavin's initial interest in mathematical finance applications and without contribution from our direct collaborators and all those who explored and tested the Fourier estimation theory in their research. We are indebted to all them.

Particular thanks go to the organizers and participants of the Conference on Modeling High-Frequency Data in Finance at Steven Institute of Technology in 2010, for their stimulating feedback which led to the survey by Mancino and Sanfelici (2011c) that is the germ of this book project. Moreover, we wish to thank Joseph Teichmann and Christa Cuchiero who kindly contributed to Section 4.3 with some of their codes and to Fabrizio Laurini for his useful comments.

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Contents

Chapter 1
Introduction

Labitur occulte fallitque volatilis aetas
(Ovidio, Metamorfosi, Liber X v. 519–520)

Measurement of the volatility/covariance of financial-asset returns plays a central role in many issues in finance, e.g., risk and investment management, hedging strategies, forecasting. In connection with financial markets the word *volatility* is usually associated with the concepts of *risk* and *opportunity*, thus referring to a measure (as well as a feeling) of the movements and uncertainty in the markets. As a matter of fact, the constant-volatility assumption prescribed by the Black & Scholes model (Black and Scholes (1973)) does not account for some stylized facts such as variance heteroscedasticity, predictability, volatility smile, covariance between asset returns and volatility (the so-called leverage effect). Therefore, a wide set of time-dependent (stochastic) volatility models have been proposed to model asset-price evolution and to price options coherently with this evidence. Nevertheless, the volatility process is unobservable and its latency leads to the difficult task of developing efficient methods to measure it.

1.1 Implied and Historical Volatility

To measure volatility, both forward- and backward-looking methods are adopted: the *implied* and the *historical volatility* approaches. The former infers volatility levels by using options markets and has been privileged by practitioners for the purpose of forecasting. The implied volatility of an option is the measure of volatility that, when used in an option-valuation model, equates the theoretical value and the market value. If option pricing models are valid, implied volatilities express the market expectation about future volatility. The main reason for using implied volatility is the assumption that the market as a whole "may know some things about the future volatility in the stock that we don't know," with Black (1975). Interested readers will

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find empirical and theoretical studies in Rubinstein (1994), Dupire (1994), Derman and Kani (1994) along with many others. More recently, a model-free measure of implied volatility that equals the market risk-neutral expectation of the total return variation has been introduced (see Britten-Jones and Neuberger (2000), Bollerslev et al. (2009, 2011)). On the contrary, the historical volatility measure is based on the magnitude of recent (past) moves of the prices, namely the (annualized) standard deviation of the log-returns. Volatility can be computed through *parametric* or *nonparametric* methods (see, for instance, the insightful review by Andersen et al. (2010)). In the first case, the expected volatility is modeled through a functional form of market or latent variables. In contrast, nonparametric methods address the computation of historical volatility without assuming any functional form of the volatility. The method studied in this book belongs to the second class. Finally, filtering methods have been applied to infer the volatility as well as its empirical distribution from historical asset-price observations, obtaining predictive distributions for multistep forecasts of volatility (among many, relevant contributions are Jacquier et al. (1994), Cvitanic et al. (2006), Chronopoulou and Viens (2012)).

1.2 High-Frequency Data

In the stochastic modeling of financial markets, the instantaneous volatility is described by the diffusion coefficient of a continuous time process. Measuring the diffusion coefficient from the observed asset prices is a challenging task, since data are not available continuously, but only on a discrete time grid. As volatility changes over time, its computation through nonparametric methods concentrates on a small time window (a day, a week), and high-frequency data are employed. In fact, the recent availability of time observations for all quotes and transactions, named *ultra-high-frequency* data by Engle (2000), has improved the capability of computing volatility efficiently, giving us new fundamental instruments and additional information about variation in return volatility, i.e., in the second moments of returns. Early recognition of this potential gain endowed by the use of high-frequency data has been noted by Nelson (1990, 1991), Andersen and Bollerslev (1998). Sophisticated technological tools and computer algorithms to rapidly trade securities have contributed to make high-frequency trading strategies more widely used by practitioners. Whereas at the turn of the twenty-first century, high-frequency trades had an execution time of several seconds, this had decreased to milliseconds and even microseconds by 2010.

At the same time, this fact poses new challenges to researchers both from the empirical and the theoretical sides, as observed early on by O'Hara (1995), Hasbrouck (1996), Goodhart and O'Hara (1997). In fact, the behavior of observed asset prices departs from what is prescribed by theoretical models (frictionless price), being affected by *noise microstructure* effects deriving from bid-ask bounce, infrequent trading, and price discreteness, among others. Furthermore, when computing covariances between returns recorded at the highest available observation frequency, returns are obviously asynchronous across different assets. Thus, the estimation of covariances suffers from a downward bias as the sampling interval is reduced (known as the *Epps effect* by Epps (1979)).

Most often, all these sources of microstructure effects are modeled as a nuisance component, in the form of additive noise components or rounding errors; this is the main approach followed in the present book. However, a very recent line of research on high-frequency data pursues a more modeling-based approach. Cartea and Jaimungal (2015) describe a model of the limit order book where agents solve a combined optimal stopping and control problem. Kercheval and Zhang (2015) propose a machine learning framework to capture the dynamics of limit order books. Other examples include the artificial "zero-intelligence" order-driven market model of Gatheral and Oomen (2010) and the Markovian queueing model of Cont and De Larrard (2013), proposing simple and tractable stochastic models for the dynamics of a limit order book in which orders to buy and sell are centralized and executed against the best available offers in the limit order book. These equilibrium models of limit order markets provide a glimpse into the dynamics of supply and demand and their role in price formation and are an attempt to describe the complex mechanisms producing microstructure effects.

1.3 Fourier Analysis for Volatility Measurement

Considering these specific characteristics of high-frequency data, a number of alternative volatility/covariance estimators have been proposed in the academic literature in the last twenty years. Most of them rely on the *quadratic covariation* formula, a classical result essentially due to Wiener, which permits the volatility in a time interval (integrated volatility) to be recovered from the observed asset prices. The *realized volatility-quadratic variation* estimators have been intensively studied and used for financial-econometrics purposes in a series of papers, and modifications of the realized volatility have been proposed to correct the bias due to microstructure noise (see Aït-Sahalia and Jacod (2014) for an updated presentation).

This book is devoted to studying an alternative nonparametric method originally proposed in Malliavin and Mancino (2002a) to compute the instantaneous multivariate volatility based on Fourier series. Owing to the book by Fourier (1822), Fourier analysis has been used in many fields because it allows one to represent a set of data as a sum of sinusoidal functions. A function of time, which is called *the signal*, is decomposed into the frequencies that constitute it. Therefore, the Fourier transform is frequently called the *frequency domain representation* of the original signal. Fourier analysis has been extensively applied to inference of processes in time-series analysis. However, these methods mainly hinge on the availability of a very long series of data and on the stationary or ergodic properties which are crucial for long time asymptotics. This fact contrasts with the approach of high-frequency data, where a finite horizon is considered and infill asymptotics (i.e., the time between two observations goes to zero) is performed, which exploits tick-by-tick data.

On the other hand, the underlying financial models fail to have stationary or ergodic properties, unlike the usual time series asymptotics prescribes. Regarding this point, the Fourier-Malliavin estimation approach does not assume any long range stationary condition as usually done in the statistical study of time series when using the ergodic theorem to compute a spectral measure or some other invariant from a single realization of the process. However, the fact that we need to construct an estimator of the desired quantity using only a single realization of the process is peculiar to financial experiments because, in contrast to other physical experiments, averaging the quantities obtained in each time window, e.g. one day, is meaningless.

Chapter 2
A First Glance at Fourier Method

Before tackling the estimation procedure in details, this chapter introduces the basic idea underlying the Fourier-Malliavin method, that is a general identity relating the Fourier transform of the (multivariate) volatility function with the Fourier transform of the log-returns. Moreover, some peculiar features of the method are briefly presented which will be more deeply addressed in the next chapters. The method has been originally proposed by Malliavin and Mancino (2002a) to estimate instantaneous multivariate volatilities from high-frequency observations of diffusion processes in a non-parametric way and without any stationarity assumption. The authors aimed at overcoming some difficulties arising from the application of the quadratic variation formula in the commonly used realized covariation methods with financial data.

2.1 Main Convolution Formula

The very first idea which led to the construction of the Fourier-Malliavin volatility estimator consists in the mathematical link between the Fourier transform of the *observed* asset prices and the Fourier transform of the *unobservable* volatility process. This section starts with an illustration of this main result.

From a theoretical viewpoint, suppose for the moment that the prices of d assets $p(t) = (p^1(t), \dots, p^d(t))$ are observed in *continuous time* over a time interval $[0,T]$ and described by d continuous processes satisfying the following Itô stochastic differential equations[1]

$$dp^j(t) = \sum_{k=1}^{l} \sigma_k^j(t)\, dW^k(t) + b^j(t)\, dt, \quad j = 1, \dots, d, \tag{2.1}$$

[1] The reader eventually unfamiliar with these dynamics for the price process can find a short introduction in the Appendix A.1.1.

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where $W = (W^1, \dots, W^l)$ are independent Brownian motions and σ_k^j and b^j are random processes satisfying mild regularity conditions which will be specified in the following sections. From the representation (2.1) the *(time-dependent) volatility*[2] *matrix* is defined as the matrix $\Sigma(t)$, whose (stochastic) entries are

$$\Sigma^{i,j}(t) = \sum_{k=1}^{l} \sigma_k^i(t)\sigma_k^j(t), \quad i,j = 1,\dots,d. \tag{2.2}$$

For both functions, the asset return and the volatility matrix, the Fourier transform can be defined as follows, for any integer k and $i,j = 1,\dots,d$, let

$$\mathscr{F}(dp^i)(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\,dp(t)$$

and

$$\mathscr{F}(\Sigma^{i,j})(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\,\Sigma^{i,j}(t)dt\,. \tag{2.3}$$

Note that by rescaling the unit of time[3] we can always reduce ourselves to the case where the time window $[0,T]$ becomes $[0,2\pi]$.

> *First Step*: for any integer k, compute the Fourier coefficients $\mathscr{F}(\Sigma^{i,j})(k)$ of the spot volatilities $\Sigma^{i,j}(t)$ by means of the Fourier coefficients $\mathscr{F}(dp^i)(k)$ of the price process $p(t)$.

Theorem 2.1. *Consider a process $p(t)$ satisfying (2.1). Then, for any $i,j = 1,\dots,d$, it holds*

$$\frac{1}{2\pi}\mathscr{F}(\Sigma^{i,j}) = \mathscr{F}(dp^i) * \mathscr{F}(dp^j), \tag{2.4}$$

where the convolution product *which appears in (2.4) is defined as follows: for any i,j and for all integers k*

$$(\mathscr{F}(dp^i) * \mathscr{F}(dp^j))(k) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|s|\le N} \mathscr{F}(dp^i)(s)\mathscr{F}(dp^j)(k-s). \tag{2.5}$$

The convergence of the convolution product (2.5) is attained in probability.[4]

We give a sketch of the proof that can be found in Malliavin and Mancino (2009). A preliminary step shows that the drift $b := (b^1, \dots, b^d)$ of (2.1) gives no contribution to the formula (2.4). Therefore, we can assume $b = 0$. For any integer k and $i = 1,\dots,d$, we set

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[2] In the econometric literature the term *volatility* is often used as a synonym of *variance*.

[3] The analogous expressions for the Fourier transforms on $[0,T]$ are given in Appendix A.2, which contains a short introduction to Fourier analysis.

[4] See Definition A.4.

$$\Gamma_k^i(t) := \frac{1}{2\pi}\int_0^t e^{-ik\tau}\, dp^i(\tau).$$

Then, by definition it holds $\Gamma_k^i(2\pi) = \mathscr{F}(dp^i)(k)$. For any integer $N \geq 1$ and any $|k| \leq N$, define

$$\gamma_k^{i,j}(N) := \frac{1}{2N+1}\sum_{|s|\leq N} \Gamma_s^i(2\pi)\Gamma_{k-s}^j(2\pi). \tag{2.6}$$

Note that the limit of (2.6) for $N \to \infty$ is equal to (2.5). By Itô formula (A.8), it follows that

$$\gamma_k^{i,j}(N) = \frac{1}{2\pi}\mathscr{F}(\Sigma^{i,j})(k) + R_N^{i,j}(k), \tag{2.7}$$

where

$$R_N^{i,j}(k) := \frac{1}{2N+1}\int_0^{2\pi} \sum_{|s|\leq N} \Gamma_s^i(t)d\Gamma_{k-s}^j(t) + \Gamma_{k-s}^j(t)d\Gamma_s^i(t)\,.$$

Therefore, the result holds true if we prove that, for any fixed k, $R_N^{i,j}(k)$ converges to 0 in probability, as $N \to \infty$. By writing $R_N^{i,j}(k)$ more explicitly, it is evident that it is equal to the sum of two analogous terms, each having the following expression

$$\frac{1}{(2\pi)^2}\int_0^{2\pi} dp^j(t_2)\int_0^{t_2} e^{ikt_1} D_N(t_1 - t_2)dp^i(t_1), \tag{2.8}$$

where $D_N(t)$ is the rescaled Dirichlet kernel

$$D_N(t) := \frac{1}{2N+1}\sum_{|s|\leq N} e^{ist} = \frac{1}{2N+1}\frac{\sin[(N+\frac{1}{2})t]}{\sin\frac{t}{2}}. \tag{2.9}$$

By Itô energy identity (A.5) and some stochastic calculus, the variance of (2.8) is proved to be less or equal to

$$C\int_0^{2\pi} D_N^2(u)du = C\,\frac{2\pi}{2N+1},$$

where C is a constant, not depending on k. For the last identity, see, e.g., Malliavin (1995). Therefore, letting $N \to \infty$ in (2.7), the proof is completed. □

As soon as all the Fourier coefficients of the volatility matrix's entries have been computed, it suffices to apply an inversion formula to obtain the time-dependent volatility function.

> *Second Step*: Reconstruct the spot volatility matrix $\Sigma(t)$ using the Fourier-Fejér inversion formula.

The reconstruction of the stochastic function of time $\Sigma^{i,j}(t)$ from its Fourier coefficients can be obtained as follows: for $i = 1,\ldots,d$ and $|s| \leq 2N$, compute the Fourier coefficients of prices $\mathscr{F}(dp^i)(s)$ and, for any $|k| \leq N$, $i,j = 1,\ldots,d$, define

$$\mathscr{F}(\Sigma_N^{i,j})(k) := \frac{2\pi}{2N+1} \sum_{|s|\leq N} \mathscr{F}(dp^i)(s)\,\mathscr{F}(dp^j)(k-s). \tag{2.10}$$

If the volatility matrix has continuous paths, namely the function $t \to \Sigma^{i,j}(t)$ is continuous, then the Fourier-Fejér summation (see Appendix A.2, formula (A.13)) gives almost surely[5] and uniformly in time

$$\Sigma^{i,j}(t) = \lim_{N\to\infty} \sum_{|k|<N} (1-\frac{|k|}{N})\mathscr{F}(\Sigma_N^{i,j})(k)\, e^{\mathrm{i}kt}\,, \quad \text{for all } t \in (0, 2\pi). \tag{2.11}$$

Remark 2.1. In view of the issues we are going to study, we emphasize that all the volatility Fourier coefficients (2.3) are obtained by the formula (2.4). In particular, the 0-th Fourier coefficient

$$\mathscr{F}(\Sigma^{i,j})(0) := \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{i,j}(t)dt$$

is computed. When multiplied by 2π, this coincides with a financially relevant quantity, that is the *integrated cross-volatility*.

Remark 2.2. It is possible to implement the Fourier method by expanding the volatility function $\Sigma^{i,j}(t)$ as a series of sines and cosines, as it has been originally done by Malliavin and Mancino (2002a). This result is a direct consequence of Remark A.2.

2.2 Specific Features of the Fourier Approach

In this section we highlight a few peculiar features of the Fourier estimation approach which result from the application of (2.4) and (2.11) with discretely observed asset price. These properties will be further studied throughout the book, even in comparison with other estimators.

Define the discrete analogue of the quantities introduced in Theorem 2.1. For notational simplicity, let us consider the case of two assets, which trade, respectively, on discrete grids $\{0 = t_0^j < t_1^j < \ldots < t_{n_j}^j = 2\pi\}$, with $j = 1, 2$. It is worth noting that we allow irregularly spaced observation times and even asynchronous observations across different assets, as is usually the case with real transaction prices.

For any integer k, $|k| \leq 2N$, let us define the discrete Fourier transform for each asset return

$$c_k(dp_{n_j}^j) := \frac{1}{2\pi}\sum_{l=0}^{n_j-1} e^{-\mathrm{i}kt_l^j}\,\delta_{I_l^j}(p^j), \tag{2.12}$$

where $I_l^j := [t_l^j, t_{l+1}^j[$ and $\delta_{I_l^j}(p^j) := p^j(t_{l+1}^j) - p^j(t_l^j),\ l = 0, \ldots, n_j - 1$ with $j = 1, 2$. For any $|k| \leq N$ and $i, j = 1, 2$, let us consider the discrete analogue of the convolution (2.5), given by

[5] See Definition A.5.

$$\frac{1}{2N+1}\sum_{|s|\leq N} c_s(dp^i_{n_i})c_{k-s}(dp^j_{n_j}).$$

In virtue of the identity (2.4), the last term, when multiplied by 2π, is the candidate as estimator of the k-th Fourier coefficient of $\Sigma^{i,j}$. Therefore, we define

$$c_k(\Sigma^{i,j}_{n_i,n_j,N}) := \frac{2\pi}{2N+1}\sum_{|s|\leq N} c_s(dp^i_{n_i})c_{k-s}(dp^j_{n_j}). \tag{2.13}$$

Finally, the random function of time

$$\widehat{\Sigma}^{i,j}_{n_i,n_j,N,M}(t) := \sum_{|k|\leq M}\left(1-\frac{|k|}{M}\right)c_k(\Sigma^{i,j}_{n_i,n_j,N})\,e^{\mathrm{i}kt} \tag{2.14}$$

will be called the *Fourier estimator* of the instantaneous volatility matrix $\Sigma^{i,j}(t)$.

We highlight here some particular features of the just described estimation procedure that will be extensively studied in the following chapters.

The definition of the Fourier spot volatility estimator (2.14) relies on the *integration* of the price observations rather than on a differentiation procedure.

This property is peculiar of the Fourier approach, as opposed to the realized volatility-type estimators (see the recent book by Aït-Sahalia and Jacod (2014) for a comprehensive treatment of these estimators). To be more specific, let us recall the procedure leading to the realized spot volatility-type estimators.

Consider the univariate case, that is the stochastic process p is defined by (2.1) with $d = l = 1$. Firstly, volatility is computed over finite time intervals $[0,t]$ (integrated volatility), relying upon the *quadratic variation* formula defined by

$$\langle p,p\rangle_t := \lim_{n\to\infty}\sum_{0\leq k<t2^n}\left(p((k+1)2^{-n}) - p(k2^{-n})\right)^2. \tag{2.15}$$

In fact, a classical result, essentially due to Wiener, states that the following identity holds almost surely

$$\langle p,p\rangle_t = \int_0^t \sigma^2(s)\,ds, \tag{2.16}$$

where σ^2 is the volatility function (denoted $\Sigma^{1,1}$ in the notation of (2.2)). Then, the spot volatility is derived from (2.16) by differentiation

$$\sigma^2(t) = \lim_{h\to 0}\frac{\int_0^{t+h}\sigma^2(s)ds - \int_0^t \sigma^2(s)ds}{h} = \lim_{h\to 0}\frac{\int_t^{t+h}\sigma^2(s)ds}{h}. \tag{2.17}$$

As a consequence, the realized volatility-type estimators measure the spot volatility at t as (weighted) sample averages of increasingly finer sampled squared (or absolute) returns over smaller and smaller $[t, t+h]$ intervals. The procedure involves a double asymptotics (for $n \to \infty$ and $h \to 0$) in order to perform both the numerical derivative (2.17) and the discretization procedure (2.15). This immediately raises important issues of efficiency and numerical instability, a critical point being the choice of the length of the time interval h.

> The computation of the Fourier coefficients for each asset price (2.12) and the Fourier spot cross-volatility estimator (2.14) requires neither equally spaced price observations nor preliminary synchronization of the observed data.

The Fourier estimator uses all the available data through (2.12): the possibility of using all data avoiding any preliminary manipulation of them translates into the direct use of unevenly sampled returns and even asynchronous data in the multivariate case. In fact, when recorded at the highest available observation frequency, asset returns are asynchronous across different assets. On the contrary, the realized covariance-type estimators rely on the *quadratic covariation* formula, which states that, for $i \neq j$,

$$\langle p^i, p^j \rangle_t := \lim_{n\to\infty} \sum_{0 \le k < t2^n} \left(p^i((k+1)2^{-n}) - p^i(k2^{-n})\right)\left(p^j((k+1)2^{-n}) - p^j(k2^{-n})\right) \tag{2.18}$$

is equal to

$$\int_0^t \Sigma^{i,j}(s)ds.$$

The definition of quadratic covariation (2.18) requires the data to be synchronous, thus these estimators suffer from a downward bias when applied to asynchronous intraday data.[6]

> The effectiveness of the Fourier spot volatility estimator (2.14) is obtained by balancing three parameters: the numbers of data n_j, the cutting frequency N in the convolution formula, and the number M of estimated Fourier coefficients of volatility to be used in the inversion formula. It must hold $M \le N \le n_j$, $j = 1, \ldots, d$. Choosing these parameters according with specific market characteristics guarantees the efficiency of the Fourier estimator with high-frequency market data.

[6] This behavior is known as *Epps effect*, by Epps (1979).

In Section 5 we will show that the Fourier estimator needs no correction in order to be statistically efficient and robust to some kind of market frictions at the same time. This result is due to the following properties of the Fourier estimator: on one side, it uses all available data by integration; on the other side, the high-frequency noise or short-run noise is ignored by the Fourier estimator by cutting the highest frequencies. In other words, when efficiently implemented, the Fourier estimator uses as much as possible of the available sample path without being excessively biased by the impact of market frictions.

> The Fourier estimator is defined as a global estimator, that is an estimator of the path $t \to \Sigma^{i,j}(t)$ over the whole time interval of interest.

This property is manifest in the fact that the convergence of the random function (2.14) to the covariance function holds uniformly in time. Differently from local estimators, which require the bandwidths to be tuned with the specific time t considered, in the case of the Fourier estimator it is possible to choose the cutting frequencies N and M independently of the specific instant of time, still obtaining accurate spot volatility estimates inside the whole observed time range.